The mixing and stabilization by transport noise

Bin Tang[1]

Collaborators: Dejun Luo[2] and Guohuan Zhao[2] (arXiv:2402.07484)

[1]**School of Mathematical Sciences, Peking University**

[2]Academy of Mathematics and Systems Science, CAS

The 19th Workshop on Markov Processes and Related Topics,
Fuzhou, July 23, 2024.

Outline

1. Transpot noise
2. Inviscid mixing by transport noise
3. From mixing to stabilization
4. References

Outline

1. Transpot noise
2. Inviscid mixing by transport noise
3. From mixing to stabilization
4. References

Transport noise

Consider the following stochastic equation on torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d \, (d \geq 2)$,

$$\mathrm{d}u = \lambda u \, \mathrm{d}t + \nu \Delta u \, \mathrm{d}t + \circ \mathrm{d}W_t \cdot \nabla u; \quad u(\cdot, 0) = u_0(\cdot), \qquad \text{(SHE)}$$

where $\lambda, \nu \geq 0$, $\circ \mathrm{d}$ means the Stratonovich stochastic differential and $W(x,t)$ is a divergence-free space-time noise. We call this kind of noise as transport noise.

The physical backgrounds of transport noises in stochastic fluid dynamics:

- Mechanics: (Mikulevicius and Rozovskii, 2004), (Holm, 2015) ...
- Separation of scales: (Flandoli and Pappalettera, 2021), (Flandoli and Pappalettera, 2022), (Debussche and Pappalettera, 2023) ...

Transport noise

Consider the following stochastic equation on torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d\,(d \geq 2)$,

$$\mathrm{d}u = \lambda u\,\mathrm{d}t + \nu\Delta u\,\mathrm{d}t + \circ\mathrm{d}W_t \cdot \nabla u. \quad \text{(SHE)}$$

The physical backgrounds of transport noises in stochastic fluid dynamics:

- Mechanics: (Mikulevicius and Rozovskii, 2004), (Holm, 2015) ...
- Separation of scales: (Flandoli and Pappalettera, 2021), (Flandoli and Pappalettera, 2022), (Debussche and Pappalettera, 2023) ...

Transport noise

Consider the following stochastic equation on torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d \, (d \geq 2)$,

$$\mathrm{d}u = \lambda u \, \mathrm{d}t + \nu \Delta u \, \mathrm{d}t + \circ \mathrm{d}W_t \cdot \nabla u. \tag{SHE}$$

The physical backgrounds of transport noises in stochastic fluid dynamics:

- Mechanics: (Mikulevicius and Rozovskii, 2004), (Holm, 2015) ...
- Separation of scales: (Flandoli and Pappalettera, 2021), (Flandoli and Pappalettera, 2022), (Debussche and Pappalettera, 2023) ...

This talk will specifically focus on the effects of transport noise on

- inviscid mixing ($\lambda = 0, \nu = 0$),
- stabilization ($\lambda \geq 0, \nu > 0$).

The form of noise

Consider transport noise as follows,

$$W(x,t) := \sqrt{C_d \kappa} \sum_{k\in\mathbb{Z}_0^d} \sum_{i=1}^{d-1} \theta_k \sigma_{k,i}(x) W_t^{k,i}, \qquad \text{(Noise)}$$

where $C_d = d/(d-1)$, noise intensity $\kappa > 0$, coefficient $\theta := \{\theta_k\}_k \in \ell^2(\mathbb{Z}_0^d)$ is symmetric, $\{\sigma_{k,i}\}_{k\in\mathbb{Z}_0^d, i=1,\ldots,d-1}$ are divergence-free fields:

$$\sigma_{k,i}(x) := \vec{a}_{k,i} \exp\{2\pi \mathrm{i}\, k\cdot x\}, \quad k\cdot \vec{a}_{k,i} = 0,$$

and $\{W^{k,i}\}_{k,i}$ are standard complex Brownian motions:

$$\overline{W_t^{k,i}} = W_t^{-k,i}, \quad \left[W^{k,i}, W^{l,j}\right]_t = 2t\delta_{k,-l}\delta_{i,j}.$$

Specific case: Kraichnan noise,

$$\theta_k = \frac{c_\alpha}{|k|^{(d+\alpha)/2}}, \quad \forall\, k \in \mathbb{Z}_0^d; \quad c_\alpha = \left(\sum_k \frac{1}{|k|^{(d+\alpha)}}\right)^{1/2},$$

where $\alpha > 0$. Kraichnan noise is important in studying turbulence.

The form of noise

Under the above assumptions, (SHE) has the following Itô form

$$\mathrm{d}u = \lambda u \,\mathrm{d}t + (\kappa+\nu)\Delta u \,\mathrm{d}t + \sqrt{C_d \kappa} \sum_{k\in\mathbb{Z}_0^d} \sum_{i=1}^{d-1} \theta_k \left(\vec{a}_{k,i}\cdot\nabla u\right) e_k(x) \,\mathrm{d}W_t^{k,i}.$$

The operator $\kappa\Delta$ is a "fake" dissipation term, since it is nullified by the martingale component in energy computations.

Outline

1. Transpot noise
2. Inviscid mixing by transport noise
   - Backgrounds of mixing
   - Our and previous works
   - Sketch of the Proof of mixing
   - Application: Regularized 2D-Euler equation
3. From mixing to stabilization
4. References

Transport equation

Consider the transport equation on torus $\mathbb{T}^d$,

$$\frac{\mathrm{d}}{\mathrm{d}t}u = b(x,t)\cdot\nabla u; \quad u(\cdot,0) = u_0(\cdot), \tag{TE}$$

where u_0 is mean zero and $b : \mathbb{T}^d \times [0,+\infty) \to \mathbb{R}^d$ is a divergence-free vector field. There is a corresponding Lagrangian flow $\varphi_{s,t} : \mathbb{T}^d \to \mathbb{T}^d$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{s,t}(x) = b(\varphi_{s,t}(x),t); \quad \varphi_{s,s} = \mathrm{Id}. \tag{Flow}$$

Definition of mixing

In hydrodynamic systems, vector field $b(x,t)$ is said exponential mixing, if there exists $\beta > 0$ and $c_\beta, C_\beta > 0$ such that

$$\|u(t)\|_{H^{-\beta}} \leq C_\beta\, e^{-c_\beta\, t} \|u_0\|_{H^\beta}, \quad \forall u_0 \in H^\beta(\mathbb{T}^d).$$

It is equivalent to the Lagrangian flow $\varphi_{s,t}$ satisfies

$$|\langle f \circ \varphi_{s,t}, g \rangle| \leq C_\beta\, e^{-c_\beta\,(t-s)} \|f\|_{H^\beta} \|g\|_{H^\beta}, \quad \forall f, g \in H^\beta(\mathbb{T}^d),$$

which means the strong mixing we usually use.

Remark: The inviscid mixing roughly means that the frequency of fluid towards higher.

Fields

There are a lot of works on exponential mixing. We only list some of them,

- Deterministic fields: (Iyer, Kiselev, and Xu, 2014), (Yao and Zlatoš, 2017), (Alberti, Crippa, and Mazzucato, 2019), (Elgindi and Zlatoš, 2019) ...
- Random fields: (Bedrossian, Blumenthal, and Punshon-Smith, 2022), (Pappalettera, 2022), (Cooperman, 2023) ...
- Stochastic fields: (Gess and Yaroslavtsev, 2021), (Coti Zelati, Drivas, and Gvalani, 2024), (Luo, Tang, and Zhao, 2024) ...

(Alberti, Crippa, and Mazzucato, 2019) shows that the regularity (in space) of vector field $b(x,t)$ is closely related to the exponential mixing.

Fields

There are a lot of works on exponential mixing. We only list some of them,

- Deterministic fields: (Iyer, Kiselev, and Xu, 2014), (Yao and Zlatoš, 2017), (Alberti, Crippa, and Mazzucato, 2019), (Elgindi and Zlatoš, 2019) . . .
- Random fields: (Bedrossian, Blumenthal, and Punshon-Smith, 2022), (Pappalettera, 2022), (Cooperman, 2023) . . .
- Stochastic fields: (Gess and Yaroslavtsev, 2021), (Coti Zelati, Drivas, and Gvalani, 2024), (Luo, Tang, and Zhao, 2024) . . .

(Bedrossian, Blumenthal, and Punshon-Smith, 2022): $b(x,t)$ is a solution of the Stochastic Navier-Stokes equation driven by an additive noise.

(Pappalettera, 2022): $b(x,t)$ is an Ornstein-Uhlenbeck velocity field.

(Cooperman, 2023): $b(x,t)$ is a random shear flow.

Fields

There are a lot of works on exponential mixing. We only list some of them,

- Deterministic fields: (Iyer, Kiselev, and Xu, 2014), (Yao and Zlatoš, 2017), (Alberti, Crippa, and Mazzucato, 2019), (Elgindi and Zlatoš, 2019) ...
- Random fields: (Bedrossian, Blumenthal, and Punshon-Smith, 2022), (Pappalettera, 2022), (Cooperman, 2023) ...
- Stochastic fields: (Gess and Yaroslavtsev, 2021), (Coti Zelati, Drivas, and Gvalani, 2024), (Luo, Tang, and Zhao, 2024) ...

When $b(x,t)$ is a stochastic field, then (TE) rewrites as

$$\mathrm{d}u = \circ \mathrm{d}W_t \cdot \nabla u; \quad u(\cdot, 0) = u_0(\cdot). \tag{STE}$$

Our work: Mixing by transport noise

Consider the (SHE) with $\lambda = 0$ and $\nu = 0$:

$$\mathrm{d}u = \kappa \Delta u \,\mathrm{d}t + \sqrt{C_d \kappa} \sum_{k \in \mathbb{Z}_0^d} \sum_{i=1}^{d-1} \theta_k \left(\vec{a}_{k,i} \cdot \nabla u\right) e_k(x) \,\mathrm{d}W_t^{k,i}. \qquad \text{(STE-1)}$$

Theorem 2.1 (Luo, T., Zhao, 2024^+; arXiv:2402.07484)

If initial data $u_0 \in L^2(\mathbb{T}^d)$ and noise coefficients $\{\theta_k\}_k \in \ell^2(\mathbb{Z}_0^d)$, then the solution u of Eq. (STE-1) is exponential mixing.

(i) $\forall \beta > 0$, *there exists a constant $C_{\beta,d} > 0$ such that*

$$\mathbb{E}\|u(t)\|_{H^{-\beta}}^2 \le C_{\beta,d}\, e^{-\kappa C(\theta,d)\, t/4} \|u_0\|_{L^2}^2,$$

where $C(\theta,d) \simeq_d \|\theta\|_{h^{-1}}^2 := \sum_k |k|^{-2}\theta_k^2$.

(ii) *If $\|\theta\|_{h^1}^2 := \sum_k |k|^2 \theta_k^2 < +\infty$, then for any $0 < \lambda_0/\kappa < D(\theta,d) \simeq_d \|\theta\|_{h^{-1}}^2$ and $q \in \left(0, \frac{D(\theta,d)\,\kappa}{\lambda_0} - 1\right)$, there is $C_{\kappa,\theta,d}(\omega)$ with finite q–moment such that*

$$\mathbb{P}\text{-a.s.}, \quad \|u(t)\|_{H^{-1}}^2 \le C_{\kappa,\theta,d}(\omega)\, e^{-\lambda_0 t}\, \|u_0\|_{L^2}^2, \quad \forall t \ge 0.$$

Comparsion with previous works

Consider the following equation on mainfold $\mathcal{M}$[1]

$$\mathrm{d}u = \kappa \sum_{n} \langle \sigma_k(x), \nabla u \rangle_{T\mathcal{M}} \circ \mathrm{d}W_t^{k,i}. \tag{STE-2}$$

Theorem 2.2 (Gess and Yaroslavtsev, 2021; arXiv:2104.03949.)

Let $\{\sigma_k(x)\}_k$ satify the Hörmander condition, elliptic conditions and some regularity conditions. Let the initial data $u_0 \in H^\beta(\mathcal{M}) \cap H^1(\mathcal{M})$, $\beta > 0$. Then the solution u to (STE-2) is exponential mixing:

$$\mathbb{P}\text{-a.s.}, \quad \|u(t)\|_{H^{-\beta}} \le C(\omega)\, e^{-\lambda_0 t} \, \|u_0\|_{H^\beta}, \quad \forall t \ge 0.$$

This result is coming from the exponential ergodicity of the two-point Lagrangian flow $(\varphi_t(x), \varphi_t(y))$ on off-diagonal set $\mathcal{D}^c$ and the fact

$$\mathbb{E}|\langle f \circ \varphi_t, g \rangle|^2 = \int_{\mathcal{M}\times\mathcal{M}} P_t(f \times f)(x,y)\, g(x) g(y)\, \mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

[1] $\mathcal{M}$ is a d-dimensional C^∞-smooth compact Riemannian-mainfold.

Comparsion with previous works

The divergence-free fields $\{\sigma_k(x)\}$ specifically need satisfy

- Hörmander condition: $Lie(\sigma_1, \dots, \sigma_K)(x) = T_x\mathcal{M}$.
- Elliptic condition for $(\sigma_k(\cdot), \sigma_k(\cdot))_k$ on $\mathcal{D}^c$:

$$\sum_k \left|\langle \sigma_k(x_1), v_1\rangle_{x_1} + \langle \sigma_k(x_2), v_2\rangle_{x_2}\right|^2 \geq C\big(|v_1|^2 + |v_2|^2\big),$$

  where $v_i \in T_{x_i}\mathcal{M}$ for $i = 1, 2$.
- Elliptic condition for the normalized tangent flow: ...
- Regularity conditions: there exists $\alpha \in (0, 1]$ such that

$$(x, y) \mapsto \sum_k D\sigma_k(x) \otimes D\sigma_k(y),\ (x, y) \mapsto \sum_k D^2\sigma_k(x) \otimes \sigma_k(y)$$

  are α-Hölder continuous, and ...

Note: If $\mathcal{M} = \mathbb{T}^d$, the above conditions are more difficult to verify and stronger than those in our work.

Comparsion with previous works

Consider the following equation on $\mathbb{R}^d$,

$$\mathrm{d}u = \nabla u \circ \mathrm{d}W_t. \qquad \text{(STE-3)}$$

$W(x,t)$ is a specific Kraichnan noise on $\mathbb{R}^d$, where

$$\mathbb{E}W(x,t) = 0, \quad \mathbb{E}[W(x,t), W(y,s)] = D(x-y)\,(t \wedge s),$$

with the isotropic covariance matrix D,

$$D(0) - D(r) = D_1\Big[\mathrm{Id} + \big(\frac{2}{d-1}\big)(\mathrm{Id} - r \otimes r)\Big]|r|^2.$$

Let $I_\beta(r) := \frac{1}{r^{d-2\beta}}$. Due to $\|h\|^2_{H^{-\beta}} = \langle h, I_\beta * h\rangle_{L^2}$ and

$$\big(D(0) - D(r)\big) : \nabla_r \otimes \nabla_r I_\beta = -\lambda_{d,\beta} I_\beta,$$

Itô formula gives the exponential mixing in the average.

Comparsion with previous works

Consider the following equation on $\mathbb{R}^d$,

$$\mathrm{d}u = \nabla u \circ \mathrm{d}W_t. \tag{STE-3}$$

$W(x,t)$ is a specific Kraichnan noise on $\mathbb{R}^d$, where

$$\mathbb{E}W(x,t) = 0, \quad \mathbb{E}[W(x,t), W(y,s)] = D(x-y)\,(t \wedge s),$$

with the isotropic covariance matrix D,

$$D(0) - D(r) = D_1\Big[\mathrm{Id} + \big(\frac{2}{d-1}\big)(\mathrm{Id} - r \otimes r)\Big]|r|^2.$$

Theorem 2.3 (Coti Zelati, Drivas, and Gvalani, 2024; J. Stat. Phys.)

Fix $\beta \in [0, d/2)$, the solution u to Eq. (STE-3) satisfies

$$\mathbb{E}\|u(t)\|^2_{H^{-\beta}} \le e^{-\lambda_{d,\beta}}\|u_0\|^2_{H^{-\beta}}, \quad t \ge 0,$$

where $\lambda_{d,\beta} = 2D_1\beta(d-2\beta)$.

Comparsion with previous works – Summary from results

$$\mathrm{d}u = \sum_k \sigma_k(x) \cdot \nabla u \circ \mathrm{d}W_t^k. \qquad \text{(abstract-STE)}$$

Literature & Result	Space	Initial data	Noise
A: Mixing in $\mathbb{P}$-a.s.	$\mathcal{M}$	$H^\beta(\mathcal{M}) \cap H^1(\mathcal{M})$	Hörmander conditions and some regularity conditions
B: Mixing in average	$\mathbb{R}^d$	Not specified[1]	Kraichnan noise
C: Mixing in average	$\mathbb{T}^d$	$L^2(\mathbb{T}^d)$	$\{\theta_k\}_k \in \ell^2(\mathbb{Z}_0^d)$
C: Mixing in $\mathbb{P}$-a.s.	$\mathbb{T}^d$	$L^2(\mathbb{T}^d)$	$\{\theta_k\}_k \in h^1(\mathbb{Z}_0^d)$

A : (Gess and Yaroslavtsev, 2021)

B : (Coti Zelati, Drivas, and Gvalani, 2024)

C : Our results (Luo, Tang, and Zhao, 2024)

[1] At least ensuring the well-posedness and belonging to $H^{-\beta}(\mathbb{R}^d)$.

Comparsion with previous works – Summary from methods

$$du = \sum_k \sigma_k(x) \cdot \nabla u \circ dW_t^k. \qquad \text{(abstract-STE)}$$

A (Gess and Yaroslavtsev, 2021): Studying the two-point Lagrangian flow $(\varphi_t(x), \varphi_t(y))$ and proving its exponential ergodicity.

B (Coti Zelati, Drivas, and Gvalani, 2024): Using the identity about this special Kraichnan noise. It doesn't work for torus and other noises.

Using the symmetry of noise coefficient $\theta := \{\theta\}_k$ and the particular spectrum of Δ on $\mathbb{T}^d$, we find an elementary and direct approach to the exponential mixing.

For simplicity, we consider $d = 2$. Eq. (STE-1) rewrites as

$$\mathrm{d}u = \kappa \Delta u \,\mathrm{d}t + \sqrt{2\kappa} \sum_{k\in\mathbb{Z}_0^2} \theta_k \Big(\frac{k^\perp}{|k|} \cdot \nabla u \Big) e_k(x) \,\mathrm{d}W_t^k, \quad u(0,\cdot) \in L^2(\mathbb{T}^2).$$

Then $\mathbb{E}|\hat{u}_k|^2$ satisfies the infinite dimensional ODE:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}|\hat{u}_k|^2 &= -8\pi^2 \kappa |k|^2 \mathbb{E}|\hat{u}_k|^2 + 16\pi^2 \kappa \sum_{l\in\mathbb{Z}_0^2} \theta_l^2 \frac{|k \cdot l^\perp|^2}{|l|^2} \mathbb{E}|\hat{u}_{k-l}|^2 \\ &= 16\pi^2 \kappa \sum_{l\in\mathbb{Z}_0^2} \theta_l^2 \frac{|k \cdot l^\perp|^2}{|l|^2} \Big(\mathbb{E}|\hat{u}_{k-l}|^2 - \mathbb{E}|\hat{u}_k|^2 \Big). \qquad \text{(ODE)} \end{aligned}$$

To prove exponential mixing, we only need to show that the ℓ^p norm of $Y_k := \mathbb{E}|\hat{u}_k|^2$ decays exponentially for some $p \in (1, +\infty)$. By symmetry,

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_k Y_k^p = -8\pi^2 \kappa\, p \sum_{k,l\in\mathbb{Z}_0^2} \theta_l^2 \frac{|k \cdot l^\perp|^2}{|l|^2} \big(Y_{k+l} - Y_k \big) \big(Y_{k+l}^{p-1} - Y_k^{p-1} \big).$$

Sketch of the Proof of mixing – Continue

To prove the exponential decay rate of $\sum_k Y_k^p(t)$, we need

$$\sum_k Y_k^p \leq c \sum_{k,l} \theta_l^2 \frac{|k \cdot l^\perp|^2}{|l|^2} \big(Y_{k+l} - Y_k\big)\big(Y_{k+l}^{p-1} - Y_k^{p-1}\big)$$

for some $c > 0$. We have the following inequality in one dimension

$$\sum_{n\in\mathbb{N}} a_n^p \leq \frac{2p^2}{p-1} \sum_{n\in\mathbb{N}} (n+1)^2 (a_{n+1}^{p-1} - a_n^{p-1})(a_{n+1} - a_n).$$

Let $\Gamma(k_0, n) = k_0 + \left\lfloor \frac{n+1}{2} \right\rfloor l + \left\lfloor \frac{n}{2} \right\rfloor l^\perp$ with $k_0 \cdot l^\perp > 0$. Noticing

$$\frac{|\Gamma(k_0,n) \cdot l^\perp|^2}{|l|^2} = \left| \frac{k_0 \cdot l^\perp}{|l|} + \left\lfloor \frac{n}{2} \right\rfloor \right|^2 \approx \frac{(n+1)^2}{4},$$

we can decompose $\mathbb{Z}_0^2$ into countable orbits as the above form and apply the inequality to obtain the exponential decay of $\sum_k Y_k^p(t)$.

Sketch of the Proof of mixing – Continue

We give a diagram of this decomposition,

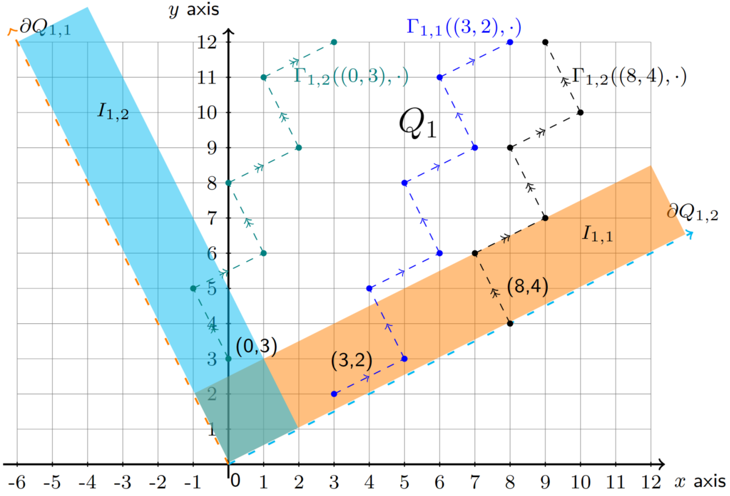


Figure: The decomposition of Q_1 for $l = (2, 1)$

Sketch of the Proof of mixing – Continue

Combining the above estimates, we obtain

$$\|Y(t)\|_{\ell^p} \le e^{-\kappa\pi^2 \frac{1}{p}(1-\frac{1}{p})\|\theta\|_{h^{-1}}^2 t}\|Y(0)\|_{\ell^p}, \quad \forall\, t > 0,$$

where $Y(t) := \{Y_k(t)\}_k$. By Hölder inequality, for $\beta > 1/2$,

$$\mathbb{E}\|u(t)\|_{H^{-\beta}}^2 = \mathbb{E}\sum_k \frac{|\hat{u}_k(t)|^2}{|2\pi k|^{2\beta}} \le \Big(\sum_k \frac{1}{|2\pi k|^{4\beta}}\Big)^{1/2}\Big(\sum_k \big(\mathbb{E}|\hat{u}_k(t)|^2\big)^2\Big)^{1/2}$$

$$\le Ce^{-\frac{\kappa\pi^2}{4}\|\theta\|_{h^{-1}}^2 t}\|Y(0)\|_{\ell^2}^4 \le Ce^{-\frac{\kappa\pi^2}{4}\|\theta\|_{h^{-1}}^2 t}\|u_0\|_{L^2}^2.$$

Similarly, the exponential mixing in average also holds for $0 < \beta \le 1/2$.

Sketch of the Proof of mixing – Continue

To obtain the exponential mixing in $\mathbb{P}$-a.s. from mixing in average, we need the Borel–Cantelli lemma and the following boundness lemma.

Lemma 2.4 (Luo, T., Zhao, 2024^+; arXiv:2402.07484)

If initial data $u_0 \in L^2(\mathbb{T}^d)$ and noise coefficient θ satisfies $\|\theta\|_{h^1}^2 < +\infty$, then the solution u to the stochastic transport equation (STE-1) satisfies

$$\mathbb{E}\Big[\sup_{t\in[0,t_0]} \|u(t)\|_{H^{-1}}^2\Big] \le 2\|u_0\|_{H^{-1}}^2,$$

where $t_0 = \big(\frac{\sqrt{11}-3}{16}\big)^2 \frac{1}{\pi^2\, d\, \kappa\, \|\theta\|_{h^1}^2} > 0$.

Application: Regularized 2D-Euler equation

We apply our results to the nonlinear equations, such as the regularized stochastic 2D Euler equation with $\alpha > 0$ on $\mathbb{T}^2$:

$$\begin{cases} \mathrm{d}w + u \cdot \nabla w \, \mathrm{d}t = \sqrt{2\kappa} \sum_k \theta_k \sigma_k \cdot \nabla w \circ \mathrm{d}W_t^k, \\ u = \operatorname{curl}^{-1}(-\Delta)^{-\alpha/2} w, \quad w(0,\cdot) \in L^2(\mathbb{T}^2), \end{cases} \tag{R-Euler}$$

where curl^{-1} is Biot–Savart operator.

Note: similar to the 2D Euler equation, the regularized equation

$$\frac{\mathrm{d}}{\mathrm{d}t} w + u \cdot \nabla w = 0, \quad u = \operatorname{curl}^{-1}(-\Delta)^{-\alpha/2} w,$$

has the following identity when w is smooth,

$$\|w(t)\|_{H^{-1-\alpha}} = \|w(0)\|_{H^{-1-\alpha}}, \quad \forall t \geq 0.$$

Application: Regularized 2D-Euler equation

We apply our results to the nonlinear equations, such as the regularized stochastic 2D Euler equation with $\alpha > 0$ on $\mathbb{T}^2$:

$$\begin{cases} \mathrm{d}w + u \cdot \nabla w \, \mathrm{d}t = \sqrt{2\kappa} \sum_k \frac{\sigma_k \cdot \nabla w}{K_\alpha |k|^{1+\alpha}} \circ \mathrm{d}W_t^k, \\ u = \mathrm{curl}^{-1}(-\Delta)^{-\alpha/2} w, \quad w(0,\cdot) \in L^2(\mathbb{T}^2), \end{cases} \tag{R-Euler}$$

where curl^{-1} is Biot–Savart operator and K_α is a normalizing constant.

Let $\tilde{W}_t^k := W_t^k - \int_0^t K_\alpha |k|^{1+\alpha} \frac{\langle u(s), \sigma_k \rangle}{\sqrt{2\kappa}} \, \mathrm{d}s$, then by Girsanov theorem, $\tilde{W}_t^k$ is a Brownian motion under $\tilde{\mathbb{P}}$. Then (R-Euler) rewrites

$$\mathrm{d}w = \kappa \Delta w \, \mathrm{d}t + \sqrt{2\kappa} \sum_k \frac{\sigma_k \cdot \nabla w}{K_\alpha |k|^{1+\alpha}} \, \mathrm{d}\tilde{W}_s^k.$$

Thus, w is exponential mixing in average with $\tilde{\mathbb{P}}$.

Application: Regularized 2D-Euler equation

We apply our results to the nonlinear equations, such as the regularized stochastic 2D Euler equation with $\alpha > 0$ on $\mathbb{T}^2$:

$$\begin{cases} \mathrm{d}w + u \cdot \nabla w \, \mathrm{d}t = \sqrt{2\kappa} \sum_k \frac{\sigma_k \cdot \nabla w}{K_\alpha |k|^{1+\alpha}} \circ \mathrm{d}W_t^k, \\ u = \mathrm{curl}^{-1} (-\Delta)^{-\alpha/2} w, \quad w(0,\cdot) \in L^2(\mathbb{T}^2), \end{cases} \tag{R-Euler}$$

where curl^{-1} is Biot–Savart operator and K_α is a normalizing constant.

Theorem 2.5 (Luo, T., Zhao, 2024^+; arXiv:2402.07484)

For fixed $R > 0$ and initial data $w_0(\cdot) := w(0,\cdot) \in L^2(\mathbb{T}^2)$. If $\|w_0\|_{L^2}^2 \le R$, then for any $\lambda > 0$, there exist a noise intensity $\kappa(\lambda, R)$ such that

$$\mathbb{E}\|w(t)\|_{H^{-1-\alpha}}^2 \le C\, e^{-\lambda t} \, \|w_0\|_{L^2}^2, \quad \forall\, t \ge 0.$$

Outline

1 Transpot noise

2 Inviscid mixing by transport noise

3 From mixing to stabilization

4 References

Backgrounds of stabilization by noise

The trivial solution $X_t \equiv 0$ of ODE

$$\dot{X}_t = A_0\, X_t, \quad X_0 \in \mathbb{R}^d,$$

is unstable if matrix $A_0 \in \mathbb{R}^{d\times d}$ has positive eigenvalues.

(Arnold, Crauel, and Wihstutz, 1983) showed that suitable linear noise induce asymptotic stability. If $\mathrm{Tr} A_0 < 0$, then there exists skew-symmetric matrices A_k such that the solution to

$$\mathrm{d}X_t = A_0\, X_t\, \mathrm{d}t + \kappa \sum_{k=1}^{d-1} A_k\, X_t \circ \mathrm{d}W_t^k$$

is $\mathbb{P}$-a.s. exponentially decays with an exponential rate arbitrarily close to $\frac{1}{d}\,\mathrm{Tr} A_0 < 0$ for κ large enough.

Backgrounds of stabilization by noise

The trivial solution $X_t \equiv 0$ of ODE

$$\dot{X}_t = A_0 \, X_t, \quad X_0 \in \mathbb{R}^d,$$

is unstable if matrix $A_0 \in \mathbb{R}^{d\times d}$ has positive eigenvalues.

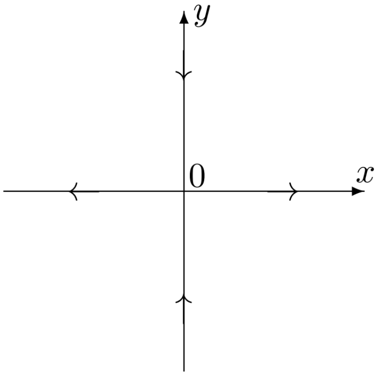
(Arnold, Crauel, and Wihstutz, 1983) showed that suitable linear noise induce asymptotic stability. If $\mathrm{Tr} A_0 < 0$, then there exists skew-symmetric matrices A_k such that the solution to

$$\mathrm{d}X_t = A_0 \, X_t \, \mathrm{d}t + \kappa \sum_{k=1}^{d-1} A_k \, X_t \circ \mathrm{d}W_t^k$$

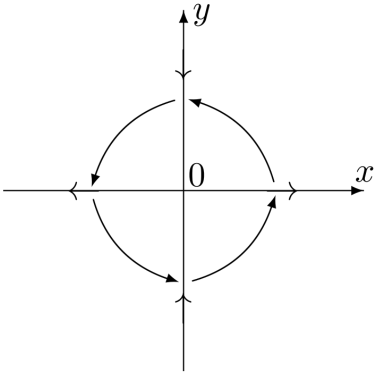
is $\mathbb{P}$-a.s. exponentially decays with an exponential rate arbitrarily close to $\frac{1}{d} \, \mathrm{Tr} A_0 < 0$ for κ large enough.

Backgrounds of stabilization by noise

Consider $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ and $\mathrm{Tr} A_0 = -1 < 0$.



$$\dot{X}_t = A_0 \, X_t$$



$$\mathrm{d}X_t = A_0 \, X_t \, \mathrm{d}t + \kappa \sum_{k=1}^{d-1} A_k \, X_t \circ \mathrm{d}W_t^k$$

Can this result be extended to the infinite dimension?

Let $\lambda \geq 0$ and $\nu > 0$. Recall Eq. (SHE) as

$$\mathrm{d}u = \lambda u \,\mathrm{d}t + \nu \Delta u \,\mathrm{d}t + \circ \mathrm{d}W_t \cdot \nabla u; \quad u(\cdot, 0) = u_0(\cdot).$$

Due to $\mathrm{Tr}(\nu\Delta + \lambda) = -\infty$, Capiński in the late 80s formualted the conjecture that under suitable nosie, the solution of (SHE) exponentially decays with an exponential rate arbitrarily close to $-\infty$.

Stabilization

Let $\lambda \geq 0$ and $\nu > 0$. Recall Eq. (SHE) as

$$\mathrm{d}u = \lambda u\,\mathrm{d}t + \nu\Delta u\,\mathrm{d}t + \circ\mathrm{d}W_t \cdot \nabla u; \quad u(\cdot,0) = u_0(\cdot).$$

Theorem 3.1 (Luo, T., Zhao, 2024^+; arXiv:2402.07484)

Given initial datum $u_0 \in L^2(\mathbb{T}^d)$ and $\theta \in \ell^2(\mathbb{Z}_0^d)$, the solution u satisfies

$$\mathbb{E}\|u(t)\|_{L^2}^2 \leq C_0\,\frac{8\pi^2\nu + D(\theta,d)\,\kappa}{2\nu}\,e^{-(-2\lambda+8\pi^2\nu+D(\theta,d)\,\kappa)\,t}\|u_0\|_{L^2}^2,$$

where $C_0 \geq \frac{1}{4\pi^2}$ and $D(\theta,d) \simeq_d \|\theta\|_{h^{-1}}^2$. There are noise parameters (κ,θ) such that $2\lambda < 8\pi^2\nu + D(\theta,d)\,\kappa$, then $\mathbb{E}\|u(t)\|_{L^2}^2$ decays exponentially.

The similar result also holds in $\mathbb{P}$-a.s. sense.

From mixing to stabilization

We follow the idea of (Gess and Yaroslavtsev, 2021) to obtain the stabilization from the exponential mixing. The key point is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{e^{2\lambda t}}{\mathbb{E}\|u(t)\|_{L^2}^2} = \frac{2\nu e^{2\lambda t}}{\left(\mathbb{E}\|u(t)\|_{L^2}^2\right)^2}\,\mathbb{E}\|u(t)\|_{H^1}^2 \geq \frac{2\nu e^{2\lambda t}}{\mathbb{E}\|u(t)\|_{H^{-1}}^2}.$$

Using the exponential mixing:

$$\mathbb{E}\|u(t)\|_{H^{-1}}^2 \leq C\, e^{-(-2\lambda+8\pi^2\nu+D(\theta,d)\,\kappa)\,t}\|u_0\|_{L^2}^2,$$

we obtain the stabilization for $\kappa > 0$ big enough.

Thank you !

References I

Alberti, Giovanni, Gianluca Crippa, and Anna L. Mazzucato (2019). "Exponential self-similar mixing by incompressible flows". In: *J. Amer. Math. Soc.* 32.2, pp. 445–490. DOI: 10.1090/jams/913.

Arnold, L., H. Crauel, and V. Wihstutz (1983). "Stabilization of linear systems by noise". In: *SIAM J. Control Optim.* 21.3, pp. 451–461. DOI: 10.1137/0321027.

Bedrossian, Jacob, Alex Blumenthal, and Samuel Punshon-Smith (2022). "Almost-sure exponential mixing of passive scalars by the stochastic Navier-Stokes equations". In: *Ann. Probab.* 50.1, pp. 241–303. DOI: 10.1214/21-aop1533.

Cooperman, William (2023). "Exponential mixing by shear flows". In: *SIAM J. Math. Anal.* 55.6, pp. 7513–7528. DOI: 10.1137/22M1513861.

Coti Zelati, Michele, Theodore D. Drivas, and Rishabh S. Gvalani (2024). "Mixing by Statistically Self-Similar Gaussian Random Fields". In: *J. Stat. Phys.* 191.5, p. 61. DOI: 10/gtxtt6.

References II

Debussche, Arnaud and Umberto Pappalettera (2023). *Second order perturbation theory of two-scale systems in fluid dynamics*.

Elgindi, Tarek M. and Andrej Zlatoš (2019). "Universal mixers in all dimensions". In: *Adv. Math.* 356, pp. 106807, 33. DOI: 10.1016/j.aim.2019.106807.

Flandoli, Franco and Umberto Pappalettera (2021). "2D Euler equations with Stratonovich transport noise as a large-scale stochastic model reduction". In: *J. Nonlinear Sci.* 31.1, Paper No. 24, 38. DOI: 10.1007/s00332-021-09681-w.

— (2022). "From additive to transport noise in 2D fluid dynamics". In: *Stoch. Partial Differ. Equ. Anal. Comput.* 10.3, pp. 964–1004. DOI: 10.1007/s40072-022-00249-7.

Gess, Benjamin and Ivan Yaroslavtsev (2021). "Stabilization by Transport Noise and Enhanced Dissipation in the Kraichnan Model". In: *arXiv: 2104.03949*.

References III

Holm, Darryl D. (2015). "Variational principles for stochastic fluid dynamics". In: *Proc. A.* 471.2176, pp. 20140963, 19. DOI: 10.1098/rspa.2014.0963.

Iyer, Gautam, Alexander Kiselev, and Xiaoqian Xu (2014). "Lower bounds on the mix norm of passive scalars advected by incompressible enstrophy-constrained flows". In: *Nonlinearity* 27.5, pp. 973–985. DOI: 10.1088/0951-7715/27/5/973.

Luo, Dejun, Bin Tang, and Guohuan Zhao (2024). "An Elementary Approach to Mixing and Dissipation Enhancement by Transport Noise". In: *arXiv:2402.07484.* arXiv:2402.07484.

Mikulevicius, R. and B. L. Rozovskii (2004). "Stochastic Navier-Stokes equations for turbulent flows". In: *SIAM J. Math. Anal.* 35.5, pp. 1250–1310. DOI: 10.1137/S0036141002409167.

References IV

Pappalettera, Umberto (2022). "Quantitative mixing and dissipation enhancement property of Ornstein-Uhlenbeck flow". In: *Comm. Partial Differential Equations* 47.12, pp. 2309–2340. DOI: 10.1080/03605302.2022.2122835.

Yao, Yao and Andrej Zlatoš (2017). "Mixing and un-mixing by incompressible flows". In: *J. Eur. Math. Soc. (JEMS)* 19.7, pp. 1911–1948. DOI: 10.4171/JEMS/709.